PRACTICAL ALGORITHM FOR SOLVING THE CUBIC EQUATION

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The Practical Algorithm tabulated below solves the cubic equation using real-number calculations only. The algorithm applies the *Numerical Recipes* approach by Press, et al.[1, §5.6] for cubic equations with one real solution. *Numerical Recipes* modifies the original Cardano formula[2, Chapter XI] to avoid Cardano's relatively large solution error for some cases due to round-off. Even so, *Numerical Recipes* addresses only a portion of Cardano's problematic cases. Our algorithm addresses the remaining cases as well by including a formula by La Porte as described by Blinn[3, p14]. The algorithm applies Viète's trigonometric method[4] for cubic equations that have three real solutions. (See also this website's paper: "Round-Off Error Mitigation in Solving Cubic and Quartic Equations", https://quarticequations.com/Round_Off%20Error.pdf.) I am grateful to my correspondent Vadym Koliada for bringing to my attention both the *Numerical Recipes* approach and the La Porte formula.

The algorithm inputs are three real coefficients a_2, a_1, and a_0, and the outputs are the three values z_1, z_2, and z_3 such that

$$z^3 + a_2 z^2 + a_1 z + a_0 = (z - z_1)(z - z_2)(z - z_3) \quad \text{for all } z.$$

The outputs are thus the three solutions of the general cubic equation

$$z_n^3 + a_2 z_n^2 + a_1 z_n + a_0 = 0, \qquad n = 1, 2, 3. \tag{1}$$

FIGURE 1 PRACTICAL ALGORITHM FOR SOLVING THE CUBIC EQUATION

<u>Given</u>: Real coefficients a_2, a_1, and a_0,

<u>Find</u>: z_1, $z_2 = x_2 + iy_2$, and $z_3 = x_3 + iy_3$ such that $z^3 + a_2 z^2 + a_1 z + a_0 = (z - z_1)(z - z_2)(z - z_3)$ for all z.

<u>Calculate q and r</u>: $q = \frac{a_1}{3} - \frac{a_2^2}{9}$ $\qquad r = \frac{a_1 a_2 - 3a_0}{6} - \frac{a_2^3}{27}$

Case: 1: $r^2 + q^3 > 0$ ⇔ Only One Real Solution (*Numerical Recipes*)

$$A = \left(|r| + \sqrt{r^2 + q^3}\right)^{1/3}$$

for $r \geq 0$	for $r < 0$
$u = A, \quad v = -q/A$	$v = -A, \quad u = q/A$
for $q \leq 0$, $\ t_1 = u + v$ (*Numerical Recipes*)	for $q > 0$, (La Porte) $t_1 = \frac{2r}{u^2 + v^2 + q}$

$$z_1 = t_1 - \frac{a_2}{3} \qquad x_2 = x_3 = -\frac{t_1}{2} - \frac{a_2}{3}$$

$$y_2 = -y_3 = \frac{\sqrt{3}}{2}(u - v)$$

$$z_2 = x_2 + iy_2 \qquad z_3 = x_2 - iy_2$$

Case 2: $r^2 + q^3 \leq 0$ ⇔ Three Real Solutions (Viète)

$$\theta = \begin{cases} 0 & \text{if } q = 0 \\ \mathrm{Cos}^{-1}[r/(-q)^{3/2}] & \text{if } q < 0 \end{cases} \qquad 0 \leq \theta \leq \pi$$

$$\phi_1 = \theta/3 \qquad \phi_2 = \phi_1 - 2\pi/3 \qquad \phi_3 = \phi_1 + 2\pi/3$$

$$z_1 = 2\sqrt{-q}\,\cos\phi_1 - a_2/3$$

$$z_2 = x_2 = 2\sqrt{-q}\,\cos\phi_2 - a_2/3 \qquad y_2 = 0$$

$$z_3 = x_3 = 2\sqrt{-q}\,\cos\phi_3 - a_2/3 \qquad y_3 = 0$$

$$z_3 \leq z_2 \leq z_1$$

Solution z_1 is defined as the greatest real solution. Solutions $z_2 = x_2 + iy_2$ and $z_3 = x_3 + iy_3$ are either real numbers ($y_2 = y_3 = 0$) or a complex conjugate pair ($x_2 = x_3$, $y_2 = -y_3$).

The algorithm converts the general cubic equation (1) to an equivalent *depressed cubic equation* with no quadratic term:

$$t_n^3 + 3q\,t_n - 2r = 0, \qquad n = 1, 2, 3. \qquad (2)$$

The real values q and r are calculated from coefficients a_2, a_1, and a_0 as shown in the algorithm. The depressed solutions t_n are related to the general solutions z_n by

$$t_n = z_n + a_2/3 \quad \Leftrightarrow \quad z_n = t_n - a_2/3, \qquad n = 1, 2, 3.$$

In this paper, all angle values are in radian measure, and the radical $\sqrt{}$ denotes the principal square root. The principal square root of a positive real number is the positive square root. The principal square root of a negative real number is the positive imaginary square root. If z is complex with modulus r and argument ϕ such that $-\pi < \phi \le \pi$, then $z = re^{i\phi}$ and the principal square root is $\sqrt{z} = \sqrt{r}\, e^{i\phi/2}$.

CHECKING THE SOLUTIONS

The set of calculated solutions z_1, $z_2 = x_2 + iy_2$, and $z_3 = x_3 + iy_3$ of the cubic equation can be checked against the requirement that

$$z^3 + a_2 z^2 + a_1 z + a_0 = (z - z_1)(z - z_2)(z - z_3) \text{ for all z.}$$

Expand and simplify the right side of this equation, and then equate each coefficient to the corresponding coefficient on the left side to obtain

$$a_2 = -(z_1 + z_2 + z_3) \qquad a_1 = z_1z_2 + z_1z_3 + z_2z_3 \qquad a_0 = -z_1z_2z_3.$$

Whether z_2 and z_3 are real ($y_2 = y_3 = 0$) or complex conjugates ($x_2 = x_3$, $y_2 = -y_3 > 0$), their sum is $z_2 + z_3 = x_2 + x_3$ and their product is $z_2z_3 = x_2x_3 + y_2^2$. Therefore, we have:

$$\boxed{a_2 = -(z_1 + x_2 + x_3) \qquad a_1 = z_1(x_2+x_3) + x_2x_3 + y_2^2 \qquad a_0 = -z_1(x_2x_3+y_2^2).}$$

Valid solutions must reproduce the input coefficients according to these check equations.

CARDANO'S SOLUTION

The above algorithm for solving the cubic equation is based on the earliest algebraic solution: the one described by Girolamo Cardano in Chapter XI of his 1545 book *Ars Magna*[2]. Cardano begins the chapter by crediting Scipio Ferro with discovery of the solution.

> Scipio Ferro of Bologna well-nigh thirty years ago discovered this rule and handed it on to Antonio Maria Fior of Venice, whose contest with Niccolò Tartaglia of Brescia gave Niccolò occasion to discover it. He [Tartaglia] gave it to me in response to my entreaties, though withholding the demonstration. Armed with this assistance, I sought out its demonstration in [various] forms. This was very difficult. My version of it follows.

Cardano uses geometry to solve the special case $t_1^3 + 3q\,t_1 = 2r$ where q and r are positive, $r^2 + q^3 \ge 0$, and only the real solution t_1 is found:

$$t_1 = \left(\sqrt{r^2+q^3}+r\right)^{1/3} - \left(\sqrt{r^2+q^3}-r\right)^{1/3}, \qquad r^2+q^3 \geq 0. \tag{3}$$

Cardano's Solution for the Depressed Cubic Equation: $t_1^3 + 3q\,t_1 - 2r = 0$

We show later that the modern derivation of Cardano's solution for t_1 easily extends to the remaining two solutions as well.

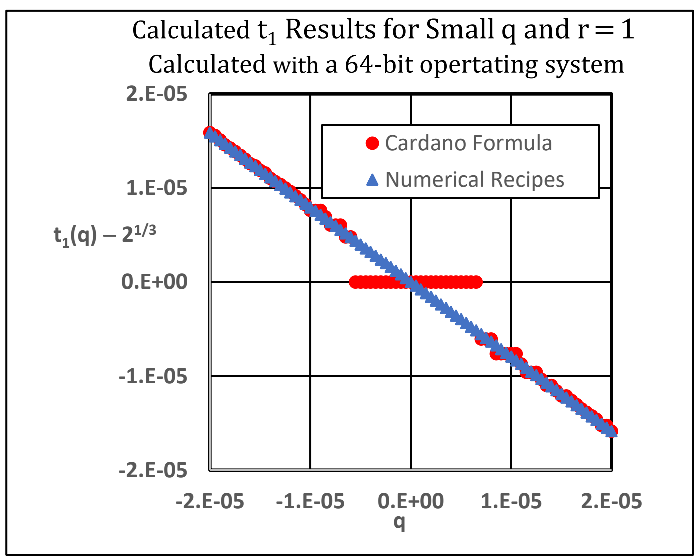
Cardano makes it clear in his book that he understands negative numbers but his audience does not. He therefore accommodates each new sign change in q or r by creating a new special case and rearranging the equation $t_1^3 + 3q\,t_1 = 2r$ so that it contains only nonnegative numbers. He uses transformations for some special cases that include a quadratic term, but he does not address the general cubic equation. In particular, he does not address the case $r^2 + q^3 < 0$, for which there are three real solutions.

For the case $r^2 + q^3 > 0$, the tabulated Practical Algorithm above uses the solution from *Numerical Recipes* rather than Cardano's solution because the Cardano produces relatively large solution error due to round-off when $0 < |q^3| << r^2$. To examine this situation, consider Cardano's formula for t_1 with r set equal to 1.

$$t_1(q) = \left(\sqrt{1+q^3}+1\right)^{1/3} - \left(\sqrt{1+q^3}-1\right)^{1/3}$$

We want to examine cases in which $|q^3| << 1$. For $q = 0$, we have $t_1(0) = 2^{1/3}$. Because $t_1(q)$ is a function of $1 + q^3$, a computer using Cardano cannot calculate a t_1 deviation from $t_1(0)$ unless $|q^3|$ exceeds the computer's precision limit. This limit is on the order of 10^{-16} for the 64-bit operating system we use below. For $|q^3|$ to exceed 10^{-16}, $|q|$ must exceed $10^{-16/3}$ or about 10^{-6} to 10^{-5}. For smaller $|q|$ values, calculated $t_1(q)$ is stuck at $t_1(0) = 2^{1/3}$.

The figure demonstrates this situation graphically. It shows that using Cardano, the computer cannot detect $t_1(q)$ deviations from $t_1(0)$ for small values of q. By contrast, *Numerical Recipes* produces accurate results for small q as expressed below:



$$t_1(q) - 2^{1/3} \cong \left[\left.\frac{dt_1(q)}{dq}\right|_{q=0}\right] q \;=\; -2^{-1/3}\,q \quad \text{for} \quad |q| << 1. \qquad \text{See Appendix A.}$$

To avoid the Cardano shortcoming, an earlier version of this paper recommended the All-Trigonometric algorithm[5, pp 174-5], [6] described in Appendix B. *Numerical Recipes* provides a much simpler algorithm than the All-Trigonometric and is more accurate than both Cardano and the All-Trigonometric.

ALGORITHM DERIVATION

Derivation of the Practical Algorithm consists of four parts. The first is derivation of Cardano's algorithm, which extends equation (3) above to include all three solutions of the cubic equation. The second part converts Cardano to the *Numerical Recipes* solution for the case $r^2 + q^3 > 0$. The third part replaces the final *Numerical Recipes* result for t_1 with the La Porte formula for the case $r^2 + q^3 > 0$, $q > 0$. The La Porte formula reduces computer round-off error in t_1 when $0 < r^2 << q^3$. The fourth part extends the Cardano derivation to Viète's trigonometric method for the case $r^2 + q^3 \le 0$.

Cardano Algorithm

The Cardano algorithm requires that the general cubic equation (1) ,

$$z_n^3 + a_2 z_n^2 + a_1 z_n + a_0 = 0, \qquad n = 1, 2, 3, \qquad (1)$$

be converted to the equivalent depressed cubic equation (2) with no quadratic term:

$$t_n^3 + 3q\,t_n - 2r = 0, \qquad n = 1, 2, 3. \qquad (2)$$

Use a shift constant c and transform $z_n = t_n - c$ to convert (1) to (2).

$$z_n^3 + a_2 z_n^2 + a_1 z_n + a_0 = (t_n - c)^3 + a_2(t_n - c)^2 + a_1(t_n - c) + a_0 =$$

$$t_n^3 + (a_2 - 3c)\,t_n^2 + (a_1 - 2a_2c + 3c^2)\,t_n + a_0 - a_1c + a_2c^2 - c^3 \;=\; t_n^3 + 3q\,t_n - 2r$$

Equate corresponding coefficients on the two sides of the last equation to produce

$$c = a_2/3, \qquad q = \frac{a_1}{3} - \frac{a_2^2}{9}, \qquad r = \frac{a_1a_2 - 3a_0}{6} - \frac{a_2^3}{27}. \qquad (4)$$

Thus, z_n and t_n are related by $\quad z_n = t_n - a_2/3, \quad (5)$

and the first step in the algorithm is to calculate q and r using (4). Once the three solutions t_n of the depressed cubic equation are known, equation (5) gives the corresponding solutions z_n of the general cubic equation.

The key to solving the depressed cubic equation $t_n^3 + 3q\,t_n - 2r = 0$ is to recognize that it has the same form as a certain identity. The cube of the binomial $U_n + V_n$ is

$$(U_n + V_n)^3 = U_n^3 + 3U_n^2V_n + 3U_nV_n^2 + V_n^3.$$

By subtracting the right side from the left and rearranging terms, the identity takes the same form as the depressed cubic equation as shown in the following diagram.

$(U_n + V_n)^3$	$+\,3$	$(-U_nV_n)$		$(U_n + V_n)$	$-$	$(U_n^3 + V_n^3)$	$= 0$	Identity
t_n^3	$+\,3$	q		t_n	$-$	$2r$	$= 0$	Depressed Cubic Equation

The identity holds for any values of U_n and V_n, even if they are complex. Therefore, if we can find values of U_n and V_n that satisfy the simultaneous equations

$$\boxed{U_n V_n = -q} \quad \text{and} \tag{6}$$

$$\boxed{U_n^3 + V_n^3 = 2r}\ , \tag{7}$$

$$\text{then } t_n \text{ is} \quad \boxed{t_n = U_n + V_n} \quad \text{for } n = 1, 2, 3. \tag{8}$$

The search for t_n becomes a search for U_n and V_n that simultaneously satisfy (6)and (7). These two equations are symmetric with respect to U_n and V_n. Let W^3 represent one of the pair (U_n^3, V_n^3). Then (7) shows that the other of the pair is $2r - W^3$. The cube of (6) becomes

$$U_n^3\, V_n^3 \;=\; V_n^3\, U_n^3 \;=\; W^3(\,2r - W^3\,) \;=\; -q^3$$

or

$$W^6 - 2r\,W^3 - \; q^3 \;=\; 0.$$

This quadratic equation in W^3 has solutions U_n^3 and V_n^3 given by the quadratic formula:

$$U_n^3 = r + \sqrt{r^2 + q^3} \qquad V_n^3 = r - \sqrt{r^2 + q^3}, \qquad n = 1, 2, 3. \tag{9}$$

These values of U_n^3 and V_n^3 satisfy the requirement of (7): $U_n^3 + V_n^3 = 2r$. Because q and r are real numbers, Equations (7) and (9) each imply that U_n^3 and V_n^3 are real or they form a complex conjugate pair.

The numbers U_n and V_n are the cube roots of U_n^3 and V_n^3. For this Cardano case, we consider only $r^2 + q^3 > 0$, so (9) shows that U_n^3 and V_n^3 are real. We define U_1 and V_1 as their real cube roots. To simplify notation in the algorithm, we replace the symbols U_1 and V_1 with u and v. The real cube roots of U_n^3 and V_n^3 in (9) provide the formulas for u and v.

$$\boxed{u \;\equiv\; U_1 \;=\; \left(r + \sqrt{r^2 + q^3}\right)^{1/3} \qquad v \;\equiv\; V_1 \;=\left(r - \sqrt{r^2 + q^3}\right)^{1/3} \qquad u > v \qquad (10)}$$

We know that $u > v$ because $\sqrt{r^2 + q^3}$ is a positive real number. Taking the product of u and v in (10) verifies that they satisfy the requirement (6): $uv = U_1V_1 = -q$. Because u and v are cube roots of U_n^3 and V_n^3, the remaining cube roots may be expressed as

$$U_2 = u\,e^{i2\pi/3} \qquad U_3 = u\,e^{-i2\pi/3} \qquad V_2 = v\,e^{-i2\pi/3} \qquad V_3 = v\,e^{i2\pi/3}.$$

Cube roots V_2 and V_3 are assigned so that so that $U_2V_2 = U_3V_3 = uv = -q$, as required.

We convert U_2, U_3, V_2, and V_3 to rectangular form and obtain solutions $t_n = U_n + V_n$ of the depressed cubic equation (2).

$$U_2 = u\left(\cos\frac{2\pi}{3} \;+ i\sin\frac{2\pi}{3}\right) = u\left(-\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right), \qquad V_2 = v\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right) = v\left(-\frac{1}{2} - i\,\frac{\sqrt{3}}{2}\right)$$

$$U_3 = u\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right) = u\left(-\frac{1}{2} - i\,\frac{\sqrt{3}}{2}\right), \qquad V_3 = v\left(\cos\frac{2\pi}{3} \;+ i\sin\frac{2\pi}{3}\right) = v\left(-\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right)$$

$$t_1 = u + v \qquad t_2 = -\frac{1}{2}(u+v) + i\,\frac{\sqrt{3}}{2}(u-v) \qquad t_3 = -\frac{1}{2}(u+v) - i\,\frac{\sqrt{3}}{2}(u-v)$$

With these formulas for t_1, t_2, and t_3, the solutions $z_n = t_n - a_2/3$ in (5) can now be calculated as follows in the Cardano algorithm.

Cardano Algorithm for $r^2 + q^3 > 0$ (one real solution and a complex conjugate pair)

Given coefficients a_2, a_1, and a_3 of the general cubic equation (1), find solutions z_2, z_1, and z_3.

$$q = \frac{a_1}{3} - \frac{a_2^2}{9}, \qquad r = \frac{a_1 a_2 - 3a_0}{6} - \frac{a_2^3}{27} \tag{4}$$

$$u = \left(r + \sqrt{r^2 + q^3}\right)^{1/3} \qquad v = \left(r - \sqrt{r^2 + q^3}\right)^{1/3} \qquad u > v \tag{10}$$

$$t_1 = u + v \tag{11}$$

$$z_1 = t_1 - \frac{a_2}{3} \qquad x_2 = x_3 = -\frac{t_1}{2} - \frac{a_2}{3} \tag{12}$$

$$y_2 = -y_3 = \frac{\sqrt{3}\,(u-v)}{2} \tag{13}$$

$$z_2 = x_2 + iy_2 \qquad z_3 = x_2 - iy_2\,. \tag{14}$$

Values u and v in (10) are real cube roots of real numbers.

Note that (10) and (11) combine with some rearranging to produce Cardano's formula (3) on page 3.

For the case $r^2 + q^3 > 0$, the Practical Algorithm, Figure 1, maintains Equations (4), (12), (13), and (14) from the Cardano Algorithm but not Equations (10) and (11).

***Numerical Recipes* Solution for Case 1: $r^2 + q^3 > 0$**

This derivation converts the Cardano equation (10) to the *Numerical Recipes* formulas for A, u, and v. Consider the condition that r^2 is much greater than $|q^3|$. A computer evaluating the radicand $r^2 + q^3$ in (10) is limited in the precision that it can maintain for q^3 when adding it to the much greater value r^2. If $|q^3|$ is sufficiently small, then $r^2 + q^3$ is evaluated simply as r^2. If $r \geq 0$, then v is calculated as zero. This zero value is obviously incorrect because v varies with q regardless of how small q becomes.

To avoid this problem, multiply and divide the expression for v by $\left(r + \sqrt{r^2 + q^3}\right)^{1/3}$:

$$v = \frac{\left(r-\sqrt{r^2+q^3}\right)^{1/3}\left(r+\sqrt{r^2+q^3}\right)^{1/3}}{\left(r+\sqrt{r^2+q^3}\right)^{1/3}} = \frac{[r^2-(r^2+q^3)]^{1/3}}{\left(r+\sqrt{r^2+q^3}\right)^{1/3}} = \frac{-q}{\left(r+\sqrt{r^2+q^3}\right)^{1/3}}, \quad r \geq 0.$$

A computer using this final expression has no trouble in accurately calculating the correct variation of v with q. *Numerical Recipes* calculates the value A defined as:

$$A = \left(|r| + \sqrt{r^2 + q^3}\right)^{1/3} > 0. \tag{15}$$

Note that A is always positive real because *Numerical Recipes* applies only to the case $r^2 + q^3 > 0$. Then with $r \geq 0$, $|r| = r$ so that Equations (10) and (15) give

$$u = A, \qquad v = -q/A \qquad (r \geq 0). \tag{16}$$

If $r < 0$, then $|r| = -r$ so that $v = -A$, and $u = \left(r + \sqrt{r^2 + q^3}\right)^{1/3} = -\left(|r| - \sqrt{r^2 + q^3}\right)^{1/3}$. Multiply and divide this u value by the expression for A in (15).

$$u = -\frac{\left(|r|-\sqrt{r^2+q^3}\right)^{1/3}\left(|r|+\sqrt{r^2+q^3}\right)^{1/3}}{\left(|r|+\sqrt{r^2+q^3}\right)^{1/3}} = -\frac{-q}{\left(|r|+\sqrt{r^2+q^3}\right)^{1/3}} = q/A, \quad r < 0$$

$$v = -A, \qquad u = q/A \qquad (r < 0) \tag{17}$$

The *Numerical Recipes* approach avoids the Cardano round-off error problem for $|q^3| << r^2$ by calculating u and v with Equations (15) to (17) rather than with Equation (10).

The Practical Algorithm, Figure 1, applies the *Numerical Recipes* Equations (15) to (17) for u and v, but uses Equation (11) $t_1 = u + v$ only when $q \leq 0$, not when $q > 0$. If $q < 0$, then u and v have the same sign. If $q = 0$, then either u or v is zero. Either way, the sum $t_1 = u + v$ is calculated accurately.

For $q > 0$, u and v have opposite signs and $|t_1| = |u + v| = ||u| - |v||$. If $r^2 << |q^3|$, then both $|u|$ and $|v|$ approach $\sqrt{q}$, the difference $|t_1| = ||u| - |v||$ approaches zero, and the calculated value of t_1 is dominated by round-off error. The La Porte formula for t_1, as described below, avoids this problem.

La Porte Formula for Case $r^2 + q^3 > 0$, $q > 0$

The La Porte formula replaces the equation $t_1 = u + v$ by noting that the sum $u + v$ is a factor of $u^3 + v^3$:

$$u^3 + v^3 = (u + v)(u^2 - uv + v^2).$$

The La Porte formula solves this identity for $t_1 = u + v$ and applies the following formulas from Equations (6), (7), and (10):

$$u^3 + v^3 = 2r \quad \text{and} \quad -uv = q.$$

The result is

$$t_1 = \frac{2r}{u^2 + v^2 + q} \tag{18}$$

The restriction $q > 0$ assures that all terms in the denominator are positive, so this formula accurately calculates t_1 for all r.

Viète's Trigonometric Method for

The Practical Algorithm applies Viète's method to Case 2: $r^2 + q^3 \leq 0 \Leftrightarrow$ Three Real Solutions on the lower right of Figure 1, page 1. The R. W. D. Nickalls paper[4] shows how

François Viète (1540–1603) solved this case with a geometric demonstration. Here we use algebra to derive Viète's trigonometric solution from equations (5) through (9):

$$z_n = t_n - a_2/3 \tag{5}$$

$$U_n V_n = -q \tag{6}$$

$$U_n^3 + V_n^3 = 2r \tag{7}$$

$$t_n = U_n + V_n \quad \text{for } n = 1, 2, 3 \tag{8}$$

$$U_n^3 = r + \sqrt{r^2 + q^3} \qquad V_n^3 = r - \sqrt{r^2 + q^3}, \qquad n = 1, 2, 3. \tag{9}$$

Note that the condition $r^2 + q^3 \le 0$ implies that $q \le 0$ and that $(-q)^3 \ge r^2$. If $q = 0$, then $r = 0$, from which (8) and (9) show that $t_n = U_n + V_n = 0$ for $n = 1, 2, 3$.

Because $r^2 + q^3 \le 0$, (9) shows that U_n^3 and V_n^3 are a complex conjugate pair:

$$U_n^3 = r + \sqrt{r^2 + q^3} \;=\; r + i\sqrt{-(r^2 + q^3)} \;=\; r + i\sqrt{(-q)^3 - r^2}$$

$$V_n^3 = r - \sqrt{r^2 + q^3} \;=\; r - i\sqrt{-(r^2 + q^3)} \;=\; r - i\sqrt{(-q)^3 - r^2}, \qquad n = 1, 2, 3.$$

The real part is r, the imaginary part is $\pm\sqrt{(-q)^3 - r^2}$, and the modulus is

$$|U_n^3| \;=\; |V_n^3| = \sqrt{r^2 + (-q)^3 - r^2} \;=\; (-q)^{3/2}.$$

The expressions for U_n^3 and V_n^3 in polar form become

$$U_n^3 = (-q)^{3/2} e^{i\theta} \quad \text{and} \quad V_n^3 (-q)^{3/2} e^{-i\theta} \tag{19}$$

where the argument θ satisfies $-1 \le \cos\theta = r/(-q)^{3/2} \le 1$, provided that $q \ne 0$. We therefore calculate θ from q and r using the principal arccosine function:

$$\theta \;=\; \text{Cos}^{-1}\left[\frac{r}{(-q)^{3/2}}\right], \qquad 0 \le \theta \le \pi \qquad (q < 0). \tag{20}$$

If $q = 0$, then the condition $r^2 + q^3 \le 0$ implies that $r = U_n^3 = V_n^3 = U_n = V_n = t_n = 0$. The value θ is irrelevant and arbitrary. For simplicity, the algorithm sets θ to zero when $q = 0$.

The three cube roots of U_n^3 and V_n^3 in (19) take the form

$$U_n = (-q)^{1/2} e^{i\phi_n} \qquad V_n = (-q)^{1/2} e^{-i\phi_n}, \qquad n = 1, 2, 3 \qquad \text{where} \tag{21}$$

$$\phi_1 = \theta/3 \qquad \phi_2 = \phi_1 - 2\pi/3 \qquad \phi_3 = \phi_1 + 2\pi/3. \tag{22}$$

Inspection shows that the formulas for U_n and V_n in (21) satisfy the requirement of (6): $U_n V_n = -q$. Equations (21), (22), and (20) show that U_n and V_n also satisfy the requirement (7) $U_n^3 + V_n^3 = 2r$:

$$U_n^3 + V_n^3 \;=\; (-q)^{3/2}\left(e^{i3\phi_n} + e^{-i3\phi_n}\right) \;=\; 2(-q)^{3/2}\cos(3\phi_n) \;=\; 2(-q)^{3/2}\cos\theta \;=\; 2r$$

The t_n are therefore given by (8) and (21):

$$t_n \;=\; U_n + V_n \;=\; (-q)^{1/2}\left(e^{i\phi_n} + e^{-i\phi_n}\right) \qquad \Rightarrow \qquad t_n \;=\; 2\sqrt{-q}\,\cos\phi_n, \;\; n = 1, 2, 3.$$

The solutions $z_n = t_n - a_2/3$ in (5) can now be calculated as

$t_1 = 2\sqrt{-q}\,\cos\phi_1$	$z_1 = t_1 - a_2/3$		(23)
$t_2 = 2\sqrt{-q}\,\cos\phi_2$	$z_2 = x_2 = t_2 - a_2/3$	$y_2 = 0$	(24)
$t_3 = 2\sqrt{-q}\,\cos\phi_3$	$z_3 = x_3 = t_3 - a_2/3$	$y_3 = 0$	(25)
	$z_1 \geq z_2 \geq z_3.$		(26)

The Viètes method in Case 2 of Figure 1employs (20) and (22) through (25) to calculate the solutions z_n when $r^2 + q^3 \leq 0$. Inequality (26) follows from the formulas for ϕ_n and t_n and from the range $[0,\pi]$ of θ in (20). That range determines the ranges of angles ϕ_1, ϕ_2, and ϕ_3 in (22) and the ranges of their cosines:

$$0 \leq \phi_1 \leq \pi/3, \quad -2\pi/3 \leq \phi_2 \leq -\pi/3, \quad 2\pi/3 \leq \phi_3 \leq \pi \quad \Rightarrow$$

$$0 \leq |\phi_1| \leq \pi/3 \leq |\phi_2| \leq 2\pi/3 \leq |\phi_3| \leq \pi \quad \Rightarrow$$

$$1 \geq \cos\phi_1 \geq 1/2 \geq \cos\phi_2 \geq -1/2 \geq \cos\phi_3 \geq -1.$$

Thus $\cos\phi_1 \geq \cos\phi_2 \geq \cos\phi_3$. Then (26) follows from (23) through (25).

The inequality information in (26) is useful when the cubic-equation solutions are applied in the companion paper on quartic equations.

This concludes the algorithm derivation.

REFERENCES

1. Press, W.H., et al., *Numerical Recipes. The Art of Scientific Computing*, 3rd Edition, 2007, Cambridge University Press, ISBN 978-0-521-88068-8, https://iate.oac.uncor.edu/~mario/materia/nr/numrec/f5-6.pdf.

2. Cardano, Girolamo, *The Rules of Algebra (Ars Magna)* [1545], translated and edited by T. Richard Witmer. 2007 reissue, Dover Publications, Inc., Mineola, NY (1993) ISBN 0-486-45873-3.

3. Blinn, James F., “How to Solve a Cubic Equation Part 5 -- Back to Numerics”, https://courses.cs.washington.edu/courses/cse590b/13au/lecture_notes/solvecubic_p5.pdf.

4. Nickalls, R. W. D., "Viète, Descartes and the cubic equation", *The Mathematical Gazette* 2006; **90** (July; No. 518), 203–208, https://pdfs.semanticscholar.org/a4ca/119b8a221fdbe2bd3f253472f932b76e673e.pdf, https://web.archive.org/web/20190503030505/https://www.cambridge.org/core/services/aop-cambridge-core/content/view/AD8AEC94661F645DC7B46781E087DA30/S0025557200179598a.pdf/div-class-title-viete-descartes-and-the-cubic-equation-div.pdf.

5. Mishina, A.P. and I.V. Proskuryakov, *Higher Algebra: Linear Algebra, Polynomials, General Algebra* [1962], translated from the Russian by Ann Swinfen, Pergamon Press, Oxford (1965).

6. *EqWorld* : Exact Solutions > Algebraic Equations and Systems of Algebraic Equations > Algebraic Equations >Cubic Equation, http://eqworld.ipmnet.ru/en/solutions/ae/ae0103.pdf.

APPENDIX A
Behavior Of $t_1(q)$ For $|q| << 1$

Given the depressed cubic equation

$$t_n^3 + 3q\,t_n - 2 = 0,$$

we desire a first-order approximation of the greatest real solution t_1 for the condition $|q| << 1$. Solution t_1 is a function of q and thus may be represented as $t_1(q)$. The first-order approximation in our special case is therefore

$$t_1(q) \cong t_1(0) + \left[\left. \frac{dt_1(q)}{dq} \right|_{q=0} \right] q, \quad \text{for } |q| << 1.$$

Our task is to find the values of $t_1(0)$ and the derivative $[dt_1/dq]_{q=0}$.

The value of $t_1(0)$ is calculated directly from the corresponding cubic equation.

$$[t_n(0)]^3 - 2 = 0 \quad \Rightarrow \quad \boxed{t_1(0) = 2^{1/3}}$$

To find the derivative $[dt_1/dq]_{q=0}$, take the derivative of the depressed cubic equation with respect to q and with n set to 1; then solve for dt_1/dq and evaluate at $q = 0$.

$$3t_1^2 \frac{dt_1}{dq} + 3t_1 + 3q\frac{dt_1}{dq} = 0 \quad \Rightarrow \quad \frac{dt_1}{dq} = -\frac{t_1}{t_1^2 + q} \quad \Rightarrow$$

$$\left. \frac{dt_1}{dq} \right|_{q=0} = -\left. \frac{1}{t_1} \right|_{q=0} = -\frac{1}{t_1(0)} = -2^{-1/3}$$

The first-order approximation of $t_1(q)$ for small q becomes

$$\boxed{t_1(q) \cong 2^{1/3} - (2^{-1/3})\,q \quad \text{for } |q| << 1.}$$

APPENDIX B
All-Trigonometric Algorithm

The All-Trigonometric algorithm for solving the cubic equation is tabulated below. The algorithm inputs are three real coefficients a_2, a_1, and a_0, and the outputs are the three values z_1, z_2, and z_3 such that

$$z^3 + a_2 z^2 + a_1 z + a_0 = (z - z_1)(z - z_2)(z - z_3) \quad \text{for all } z.$$

The outputs are thus the three solutions of the general cubic equation

$$z_n^3 + a_2 z_n^2 + a_1 z_n + a_0 = 0, \qquad n = 1, 2, 3. \tag{B-1}$$

Solution z_1 is defined as the greatest real solution. Solutions $z_2 = x_2 + iy_2$ and $z_3 = x_3 + iy_3$ are either real numbers ($y_2 = y_3 = 0$) or a complex conjugate pair ($x_2 = x_3$, $y_2 = -y_3$).

All-Trigonometric Algorithm

<u>Given</u>: Real coefficients a_2, a_1, and a_0,

<u>Find</u>: z_1, $z_2 = x_2 + iy_2$, and $z_3 = x_3 + iy_3$ such that $z^3 + a_2 z^2 + a_1 z + a_0 = (z - z_1)(z - z_2)(z - z_3)$ for all z.

<u>Calculate q and r</u>: $q = \frac{a_1}{3} - \frac{a_2^2}{9}$ $\qquad r = \frac{a_1 a_2 - 3a_0}{6} - \frac{a_2^3}{27}$

Case A: $r^2 + q^3 > 0$, $q < 0$ $\Rightarrow$ Only One Real Solution

$\gamma = \operatorname{Sin}^{-1}[(-q)^{3/2} / r], \qquad 0 < |\gamma| < \pi/2$

$\chi = \operatorname{Tan}^{-1}\{[\tan(\gamma/2)]^{1/3}\}, \qquad 0 < |\chi| < \pi/4$

$z_1 = 2\sqrt{-q}\,\csc 2\chi - a_2/3$

$x_2 = x_3 = -\sqrt{-q}\,\csc 2\chi - a_2/3$

$y_2 = -y_3 = \sqrt{-3q}\,|\cot 2\chi|$

$z_2 = x_2 + i y_2 \qquad z_3 = x_2 - i y_2$

Case B: $r^2 + q^3 > 0$, $q = 0$ $\Rightarrow$ Only One Real Solution

$z_1 = (2r)^{1/3} - a_2/3$

$x_2 = x_3 = -\frac{1}{2}(2r)^{1/3} - a_2/3$

$y_2 = -y_3 = \frac{\sqrt{3}}{2}\,|2r|^{1/3}$

$z_2 = x_2 + i y_2 \qquad z_3 = x_2 - i y_2$

Note: $(2r)^{1/3}$ and $|2r|^{1/3}$ are real cube roots of real numbers.

Case C: $r^2 + q^3 > 0$, $q > 0$ $\Rightarrow$ Only One Real Solution

$\gamma = \begin{cases} \pi/2 & \text{if } r = 0 \\ \operatorname{Tan}^{-1}(q^{3/2}/r) & \text{if } r \neq 0 \end{cases}, \qquad 0 < |\gamma| \le \pi/2$

$\chi = \operatorname{Tan}^{-1}\{[\tan(\gamma/2)]^{1/3}\}, \qquad 0 < |\chi| \le \pi/4$

$z_1 = 2\sqrt{q}\,\cot 2\chi - a_2/3$

$x_2 = x_3 = -\sqrt{q}\,\cot 2\chi - a_2/3$

$y_2 = -y_3 = \sqrt{3q}\,|\csc 2\chi|$

$z_2 = x_2 + i y_2 \qquad z_3 = x_2 - i y_2$

Case D: $r^2 + q^3 \le 0$ $\Leftrightarrow$ Three Real Solutions (Viète)

$\theta = \begin{cases} 0 & \text{if } q = 0 \\ \operatorname{Cos}^{-1}[r/(-q)^{3/2}] & \text{if } q < 0 \end{cases} \qquad 0 \le \theta \le \pi$

$\phi_1 = \theta/3 \qquad \phi_2 = \phi_1 - 2\pi/3 \qquad \phi_3 = \phi_1 + 2\pi/3$

$z_1 = 2\sqrt{-q}\,\cos\phi_1 - a_2/3$

$z_2 = x_2 = 2\sqrt{-q}\,\cos\phi_2 - a_2/3 \qquad y_2 = 0$

$z_3 = x_3 = 2\sqrt{-q}\,\cos\phi_3 - a_2/3 \qquad y_3 = 0$

$z_3 \le z_2 \le z_1$

The algorithm converts the general cubic equation (B-1) to an equivalent *depressed cubic equation* with no quadratic term:

$$t_n^3 + 3q\,t_n - 2r = 0, \qquad n = 1, 2, 3. \tag{B-2}$$

The real values q and r are calculated from coefficients a_2, a_1, and a_0 as shown in the algorithm. The depressed solutions t_n are related to the general solutions z_n by

$$t_n = z_n + a_2/3 \quad \Leftrightarrow \quad z_n = t_n - a_2/3.$$

The All-Trigonometric algorithm is an expansion of a simpler algorithm that applies the Cardano algorithm for $r^2 + q^3 > 0$ and Viète's method for $r^2 + q^3 \le 0$. Case D of the All-Trigonometric algorithm for $r^2 + q^3 \le 0$ is Viète's method without modification. Cases A, B, and C subdivide the condition $r^2 + q^3 > 0$ into three ranges of q: $q < 0$ for Case A, $q = 0$ for Case B, and $q > 0$ for Case C. The remainder of this paper derives the All-Trigonometric Cases A, B, and C from the Cardano algorithm.

Derivation of All-Trigonometric Algorithm, Cases A, B, C:

These three cases of the All-Trigonometric Algorithm derive from the Cardano algorithm for $r^2 + q^3 > 0$, which implies that there is only one real solution.

Cardano Algorithm for $r^2 + q^3 > 0$ ⇔ Only One Real Solution

$$u = \left(r + \sqrt{r^2 + q^3}\right)^{1/3} \qquad v = \left(r - \sqrt{r^2 + q^3}\right)^{1/3}, \qquad u > v \tag{B-3}$$

$$z_1 = u + v - a_2/3 \tag{B-4}$$

$$x_2 = x_3 = -\frac{u+v}{2} - \frac{a_2}{3} \qquad y_2 = -y_3 = \frac{\sqrt{3}\,(u-v)}{2} \tag{B-5}$$

$$z_2 = x_2 + iy_2 \qquad z_3 = x_2 - iy_2 \tag{B-6}$$

Note: u and v are real cube roots of real numbers.

The values $u + v$ and $u - v$ in (B-3) may be expressed as follows by using $r^{1/3}$ as a factor.

$$u + v = r^{1/3}\left[\left(1 + \tfrac{|r|}{r}\sqrt{1 + q^3/r^2}\,\right)^{1/3} + \left(1 - \tfrac{|r|}{r}\sqrt{1 + q^3/r^2}\,\right)^{1/3}\right]$$

$$u - v = r^{1/3}\left[\left(1 + \tfrac{|r|}{r}\sqrt{1 + q^3/r^2}\,\right)^{1/3} - \left(1 - \tfrac{|r|}{r}\sqrt{1 + q^3/r^2}\,\right)^{1/3}\right].$$

Regardless of the sign of r, the expressions simplify to

$$u + v = r^{1/3}\left[\left(1 + \sqrt{1 + q^3/r^2}\,\right)^{1/3} + \left(1 - \sqrt{1 + q^3/r^2}\,\right)^{1/3}\right] \tag{B-7}$$

$$u - v = \left|r^{1/3}\right|\left[\left(1 + \sqrt{1 + q^3/r^2}\,\right)^{1/3} - \left(1 - \sqrt{1 + q^3/r^2}\,\right)^{1/3}\right]. \tag{B-8}$$

Case A

Case A is defined by the condition: $r^2 + q^3 > 0$, $q < 0$. The following restrictions therefore apply:

$r^2 + q^3 > 0, q < 0 \;\Rightarrow\; 0 < (-q)^3 < r^2 \;\Rightarrow\; 0 < (-q)^3/r^2 < 1 \;\Rightarrow\; 0 < (-q)^{3/2}/|r| < 1.$ (B-9)

Define angles γ and χ as follows.

$\gamma \equiv \mathrm{Sin}^{-1}[(-q)^{3/2}/r]$,	$0 < \lvert\gamma\rvert < \pi/2$	(B-10)
$\chi \equiv \mathrm{Tan}^{-1}\{[\tan(\gamma/2)]^{1/3}\}$,	$0 < \lvert\chi\rvert < \pi/4$	(B-11)

The restriction (B-9) determines the ranges of γ and χ, and it assures that all fractions in the following expressions have non-zero denominators. Also notice that the functional relationships in (B-10) and (B-11) between $[(-q)^{3/2}/r]$, γ, and χ are continuous, odd, and monotonic increasing. Thus, γ and χ have the same sign as r.

In (B-7) and (B-8), replace the radicand $1+q^3/r^2$ with $1-(-q^3/r^2)$.

$$u + v \;=\; r^{1/3}\left\{\left[1+\sqrt{1-(-q^3/r^2)}\right]^{1/3} + \left[1-\sqrt{1-(-q^3/r^2)}\right]^{1/3}\right\}$$

$$u - v \;=\; \left|r^{1/3}\right|\left\{\left[1+\sqrt{1-(-q^3/r^2)}\right]^{1/3} - \left[1-\sqrt{1-(-q^3/r^2)}\right]^{1/3}\right\}.$$

From (B-10), replace $(-q^3/r^2)$ with $\sin^2\gamma$ so that the radicals simplify to $|\cos\gamma|$. The restriction $0 < |\gamma| < \pi/2$ implies that $0 < \cos\gamma < 1$. Thus, $|\cos\gamma| = \cos\gamma$, and the values enclosed in braces are positive. Also, from (B-10), replace the factor $r^{1/3}$ with $(-q)^{1/2}/(\sin\gamma)^{1/3}$.

$$u + v \;=\; (-q)^{1/2}\,\frac{\{(1+\cos\gamma)^{1/3} + (1-\cos\gamma)^{1/3}\}}{(\sin\gamma)^{1/3}} \;=\; (-q)^{1/2}\left\{\left(\frac{1+\cos\gamma}{\sin\gamma}\right)^{1/3} + \left(\frac{1-\cos\gamma}{\sin\gamma}\right)^{1/3}\right\}$$

$$u - v \;=\; (-q)^{1/2}\,\frac{\{(1+\cos\gamma)^{1/3} - (1-\cos\gamma)^{1/3}\}}{\left|(\sin\gamma)^{1/3}\right|} \;=\; (-q)^{1/2}\left|\left(\frac{1+\cos\gamma}{\sin\gamma}\right)^{1/3} - \left(\frac{1-\cos\gamma}{\sin\gamma}\right)^{1/3}\right|$$

Apply the trigonometric identity $\tan\frac{\gamma}{2} = \frac{\sin\gamma}{1+\cos\gamma} = \frac{1-\cos\gamma}{\sin\gamma}$ and Equation (B-11).

$$u + v \;=\; (-q)^{1/2}\left\{\frac{1}{[\tan(\gamma/2)]^{1/3}} + \frac{[\tan(\gamma/2)]^{1/3}}{1}\right\} \;=\; (-q)^{1/2}\left(\frac{1}{\tan\chi} + \frac{\tan\chi}{1}\right)$$

$$u + v \;=\; 2(-q)^{1/2}\,\frac{1+\tan^2\chi}{2\tan\chi} \qquad\qquad u - v \;=\; 2(-q)^{1/2}\left|\frac{1-\tan^2\chi}{2\tan\chi}\right|$$

Apply the trigonometric identities $\sin 2\chi = \frac{2\tan\chi}{1+\tan^2\chi}$ and $\tan 2\chi = \frac{2\tan\chi}{1-\tan^2\chi}$ respectively to the last two equations.

$$u + v \;=\; 2\sqrt{-q}\,/\sin 2\chi \;=\; 2\sqrt{-q}\,\csc 2\chi$$

$$u - v \;=\; 2\sqrt{-q}\,/\,|\tan 2\chi| \;=\; 2\sqrt{-q}\,|\cot 2\chi|$$

Substitute these expressions into (B-4), (B-5), and (B-6) from Cardano to produce the formulas for solutions z_n.

$$z_1 = 2\sqrt{-q}\ \csc 2\chi - a_2/3 \qquad \text{(B-12)}$$

$$x_2 = x_3 = -\sqrt{-q}\ \csc 2\chi - a_2/3 \qquad y_2 = -y_3 = \sqrt{-3q}\ |\cot 2\chi| \qquad \text{(B-13)}$$

$$z_2 = x_2 + iy_2 \qquad z_3 = x_2 - iy_2 \qquad \text{(B-14)}$$

The All-Trigonometric Algorithm employs equations (B-10) through (B-14) to calculate solutions z_n when $r^2 + q^3 > 0$, $q < 0$ (Case A).

<u>Case B</u>

Case B is defined by the condition: $r^2 + q^3 > 0$, $q = 0$, which implies $|r| > 0$. The depressed cubic equation (B-2) becomes $t_n^3 - 2r = 0$, and the t_n are the three cube roots of $2r$:

$$t_1 = (2r)^{1/3}, \qquad t_2 = -\tfrac{1}{2}(2r)^{1/3} + i\tfrac{\sqrt{3}}{2}|2r|^{1/3}, \qquad t_3 = -\tfrac{1}{2}(2r)^{1/3} - i\tfrac{\sqrt{3}}{2}|2r|^{1/3}.$$

The solutions $z_n = t_n - a_2/3$ for Case B are then calculated as

$$z_1 = (2r)^{1/3} - a_2/3 \qquad x_2 = x_3 = -\tfrac{1}{2}(2r)^{1/3} - a_2/3 \qquad y_2 = -y_3 = \tfrac{\sqrt{3}}{2}\ |2r|^{1/3}$$

$$z_2 = x_2 + i\,y_2 \qquad z_3 = x_2 - i\,y_2.$$

<u>Case C</u>

Case C is defined by the condition: $r^2 + q^3 > 0$, $q > 0$. The derivation is similar to that for Case A. For Case C, however, we have $q > 0$ and must accommodate the possibility that $r = 0$. The definitions of γ and χ are now

$$\gamma \equiv \begin{cases} \pi/2 & \text{if } r = 0 \\ \mathrm{Tan}^{-1}(q^{3/2}/r) & \text{if } r \neq 0 \end{cases} \qquad 0 < |\gamma| \le \pi/2, \qquad \text{(B-15)}$$

$$\chi \equiv \mathrm{Tan}^{-1}\{[\tan(\gamma/2)]^{1/3}\}, \qquad 0 < |\chi| \le \pi/4. \qquad \text{(B-16)}$$

As in Case A, the values γ and χ have the same sign as r.

The derivation assumes that $r \neq 0$, which implies that $0 < |\gamma| < \pi/2$. As we show, however, the final formulas for solutions z_n are valid for $r = 0$ as well as for $r \neq 0$.

Modify equations (B-7) and (B-8) as follows. From (B-15), replace q^3/r^2 with $\tan^2\gamma$ so that the radicals simplify to $1/|\cos\gamma|$. The restriction $0 < |\gamma| < \pi/2$ implies that $0 < \cos\gamma < 1$. Thus, $|\cos\gamma| = \cos\gamma$, and the values enclosed in brackets are positive. Also, from (B-15), replace the factor $r^{1/3}$ with $q^{1/2}/(\tan\gamma)^{1/3}$.

$$u + v = q^{1/2}\ \frac{\left[\left(1+\sqrt{1+\tan^2\gamma}\right)^{1/3} + \left(1-\sqrt{1+\tan^2\gamma}\right)^{1/3}\right]}{(\tan\gamma)^{1/3}} = q^{1/2}\ \frac{(1+1/\cos\gamma)^{1/3} + (1-1/\cos\gamma)^{1/3}}{(\tan\gamma)^{1/3}}$$

$$u - v = q^{1/2} \frac{\left[\left(1+\sqrt{1+\tan^2\gamma}\right)^{1/3} - \left(1-\sqrt{1+\tan^2\gamma}\right)^{1/3}\right]}{\left|(\tan\gamma)^{1/3}\right|} = q^{1/2} \left|\frac{(1+1/\cos\gamma)^{1/3} - (1-1/\cos\gamma)^{1/3}}{(\tan\gamma)^{1/3}}\right|$$

Multiply the numerator and the denominator by $(\cos\gamma)^{1/3}$. Then apply the trigonometric identity $\tan\frac{\gamma}{2} = \frac{\sin\gamma}{1+\cos\gamma} = \frac{1-\cos\gamma}{\sin\gamma}$ and Equation (B-16).

$$u + v = q^{1/2} \frac{(\cos\gamma+1)^{1/3} + (\cos\gamma-1)^{1/3}}{(\sin\gamma)^{1/3}} = q^{1/2}\left[\left(\frac{1+\cos\gamma}{\sin\gamma}\right)^{1/3} - \left(\frac{1-\cos\gamma}{\sin\gamma}\right)^{1/3}\right]$$

$$u + v = q^{1/2}\left[\frac{1}{[\tan(\gamma/2)]^{1/3}} - \frac{[\tan(\gamma/2)]^{1/3}}{1}\right] = q^{1/2}\left(\frac{1}{\tan\chi} - \frac{\tan\chi}{1}\right)$$

$$u + v = 2\,q^{1/2}\frac{1-\tan^2\chi}{2\tan\chi} \qquad\qquad u - v = 2\,q^{1/2}\left|\frac{1+\tan^2\chi}{2\tan\chi}\right|$$

Apply the trigonometric identities $\tan 2\chi = \frac{2\tan\chi}{1-\tan^2\chi}$ and $\sin 2\chi = \frac{2\tan\chi}{1+\tan^2\chi}$ respectively to the last two equations.

$$u + v = 2\sqrt{q}\,/\tan 2\chi = 2\sqrt{q}\,\cot 2\chi$$

$$u - v = 2\sqrt{q}\,/\,|\sin 2\chi| = 2\sqrt{q}\,|\csc 2\chi|$$

Substitute these expressions into (B-4), (B-5), and (B-6) from Cardano to produce the formulas for solutions z_n.

$$z_1 = 2\sqrt{q}\,\cot 2\chi - a_2/3 \qquad \text{(B-17)}$$

$$x_2 = x_3 = -\sqrt{q}\,\cot 2\chi - a_2/3 \qquad y_2 = -y_3 = \sqrt{3q}\,|\csc 2\chi| \qquad \text{(B-18)}$$

$$z_2 = x_2 + iy_2 \qquad z_3 = x_2 - iy_2 \qquad \text{(B-19)}$$

The All-Trigonometric Algorithm employs equations (B-15) through (B-19) to calculate the solutions z_n when $r^2 + q^3 > 0$, $q > 0$ (Case C).

The derivation of equations (B-17) through (B-19) assumed that $r \neq 0$, but these formulas also apply to $r = 0$, as now shown. The depressed cubic equation for $r = 0$ is $t_n^3 + 3q\,t_n = 0$ with depressed solutions $t_1 = 0$, $t_2 = i\sqrt{3q}$, $t_3 = -i\sqrt{3q}$. The solutions $z_n = t_n - a_2/3$ are therefore

$$z_1 = -a_2/3, \qquad z_2 = -a_2/3 + i\sqrt{3q}, \qquad z_3 = -a_2/3 - i\sqrt{3q}.$$

The algorithm equations (B-15) through (B-19) produce the same result:

$$\gamma = \pi/2, \qquad \chi = \pi/4, \qquad z_1 = -a_2/3 \qquad x_2 = x_3 = -a_2/3 \qquad y_2 = -y_3 = \sqrt{3q}$$

$$z_2 = -a_2/3 + i\sqrt{3q}, \qquad z_3 = -a_2/3 - i\sqrt{3q}.$$

Equations (B-17) through (B-19) thus apply to all r values for Case C.